A homotopy n-sphere is an oriented smooth manifold homeomorphic to S^n.(by Generalized Poincaré conjecture) For $n \geq 5$, (the h-cobordism class) of these form an abelian group Θ_n (up to diffeomorphism) with operation # (connected sum).

Theorem 1. *Kervaire-Milnor: Θ_n is finite.*

Question 1: How does $|\Theta_n|$ grows as n grows?
For some prime p, consider:

Theorem 2. *(Burklund) Whem $M \to \infty$,*

$$\sum_{n=5}^{M} \frac{\dim_{\mathbb{F}_p}(\Theta_n \otimes_{\mathbb{Z}} \mathbb{F}_p)}{M} \to \infty$$

Theorem 3. *$\pi_{n+k}(S^k)$ is independent of k for sufficiently large k.*

Denote π_n^{st} to be the n-th stable homotopy group of sphere.

Theorem 4. *(Kervaire-Milnor) There exists exact sequence*

$$P_{n+1} \to \Theta_n \to \pi_n^{st}/J_n \to P_n$$

(Where P_{n+1} is a subgroup of Θ_n, defined by any homotopy n-sphere bounds a parallelizable manifold, J_n is the image of n-th J-homomorphism)

Remark 1. *$\pi_n^{st}(X) := \varinjlim_k \pi_{k+n}(\Sigma^k X)$. is a homology theory, satisfy excision, MV sequence, etc. Indeed, $\pi_n^{st}(\mathrm{pt}) = \pi_n^{st}$.*

Theorem 5. *If X is a finite CW complex, then either:*

1. *$\pi_n^{st}(X) \otimes \mathbb{F}_p = 0$*
2. *$\mathrm{limit}_{M\to\infty} \sum_{n=5}^{M} \dfrac{\dim_{\mathbb{F}_p}(\pi_n^{st}(X) \otimes_{\mathbb{Z}} \mathbb{F}_p)}{M} \to \infty$ holds.*

(for simplicity we only discuss $p > 5$)

There is another homology theory $(\pi^{st}/p)_*$ sitting in the long exact sequence below:

$$\pi_n^{st}(X) \xrightarrow{\times p} \pi_n^{st}(X) \to (\pi^{st}/p)_n(X) \to \pi_{n-1}^{st}(X)$$

Construction 1. *(Adams/Toda) There's a natural transformation:*

$$(\pi^{st}/p)_*(X) \xrightarrow{v_1} (\pi^{st}/p)_{*+2p-2}(X)$$

making $(\pi^{st}/p)_*(X)$ an $\mathbb{F}_p[v_1]$-modules.

Theorem 6. *(Miller) $v_1^{-1}(\pi^{st}/p)_*(pt)$ is a 2-dimensional $\mathbb{F}_p[v_1, v_1^{-1}]$-vector space.*

There is another homology theory $(\pi^{st}/p, v_1)_*(X)$ sitting in the long exact sequence below:

$$\to (\pi^{st}/p)_*(X) \xrightarrow{\times v_1} (\pi^{st}/p)_{*+2p-2}(X) \to (\pi^{st}/p, v_1)_{*+2p-2}(X) \to$$

There's a natural transformation:

$$(\pi^{st}/p, v_1)_*(X) \xrightarrow{v_2} (\pi^{st}/p, v_1)_{*+2p^2-2}(X)$$

making $(\pi^{st}/p, v_1)_*(X)$ a $\mathbb{F}_p[v_2]$-module.

Conjecture 1. *$v_2^{-1}(\pi^{st}/p, v_1)_*(pt)$ is a 12-dimensional $\mathbb{F}_p[v_2, v_2^{-1}]$-vector space.*

Theorem 7. *(Burklund-H-Levy-Schlank) The conjecture above is wrong. The dimension is infinite.*

Consequence: after invert v_2 by v_2^{-1} have infinite dimension $\implies$ there are more things before inverting v_2.

Remark 2. *The conjecture comes from telescope conjecture, that indicate things about homology theory...*

Miller's theorem also prove the theory $(\pi^{st}/p)_*$ satisfy Galois descent, so ["Telescope conjecture says $(\pi^{st}/p, v_1)_*$ should behave similarly."]

Question: How to prove theorem BHLS?

Look at $v_2^{-1}(\pi^{st}/p, v_1)_*(pt) \to v_2^{-1}(\pi^{st}/p, v_1)_*(K(J_k))$, where J_k is some K-theory object. ($k >> 0$)

There's action of ψ^m on $KU_p^\wedge$, where m is a topolofical generator of $\mathbb{Z}_p^{\wedge\times}$.

Define J_k to be the homotopy fixed points of